Random-data Cauchy Problem for the Periodic Navier-Stokes Equations with Initial Data in Negative-order Sobolev Spaces*

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Abstract

In this paper we study existence of solutions of the initial-boundary value problems of the Navier-Stokes equations with a periodic boundary value condition for initial data in the Sobolev spaces $\mathcal{H}^s(\mathbb{T}^N)$ with a negative order $-1<s<0$, where $N=2,3$. By using the randomization approach of N. Burq and N. Tzvetkov, we prove that for almost all $\omega\in\Omega$, where Ω is the sample space of a probability space $(\Omega,\mathcal{A},p)$, for the randomized initial data $\vec{f}^\omega\in\mathcal{H}^s_\sigma(\mathbb{T}^N)$ with $-1<s<0$, such a problem has a unique local solution.

Keywords: Navier-Stokes equations; initial-boundary value problem; solution; randomization.

Mathematics Subject Classification: 35Q30, 76D06

1 Introduction

In this paper we study the initial-boundary value problem of the Navier-Stokes equations with a periodic boundary value condition:

$$\begin{cases} \partial_t\vec{u}-\Delta\vec{u}+(\vec{u}\cdot\nabla)\vec{u}+\nabla P=0 \quad \text{in } \mathbb{R}_+\times\mathbb{R}^N, \\ \nabla\cdot\vec{u}=0 \quad \text{in } \mathbb{R}_+\times\mathbb{R}^N, \\ \vec{u} \text{ and } P \text{ are 1-periodic in space variables}, \\ \vec{u}(0,x)=\vec{u}_0(x) \quad \text{for } x\in\mathbb{R}^N. \end{cases} \tag{1.1}$$

Here $\vec{u}=\vec{u}(t,x)=(u_1(t,x),\cdots,u_N(t,x))$ and $P=P(t,x)$ are unknown vector and scaler functions, respectively, and $\vec{u}_0=\vec{u}_0(x)=(u_1(x),\cdots,u_N(x))$ is a given vector function which is 1-periodic and satisfies the divergence free condition $\nabla\cdot\vec{u}_0=0$.

Mathematical analysis of the Navier-Stokes equations has a long history. It goes back to the beginning of the twentieth century. In [10] Leray introduced the concept of weak solutions

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and proved existence of global weak solutions associated with $L^2(\mathbb{R}^N)$ initial data by using an approximation approach and some weak compactness argument. In [7] Kato and Fujita initiated a different approach. They proved well-posedness of the initial value problem (IVP) of the Navier-Stokes equations in $\mathcal{H}^s(\mathbb{R}^N)$ for $s \geq \frac{N}{2} - 1$, i.e., (local) existence and uniqueness of solutions for $\vec{u}_0 \in \mathcal{H}^s(\mathbb{R}^N)$ $(s \geq \frac{N}{2} - 1)$ which still belongs to this space for $t > 0$ and continuously depends on the initial data $\vec{u}_0$. This approach was later extended to various other function spaces by many authors, cf. [5] and [9] for expositions and references cited therein. In particular, a famous result due to Cannone and Planchon says that the IVP of the Navier-Stokes equations is well-posed in the Besov space $B^s_{p,q}(\mathbb{R}^N)$ for $s > -1$, $p \geq \frac{N}{1+s}$ and $1 \leq q \leq \infty$ (see [4] and [11]). Another interesting result that must be mentioned is due to Koch and Tataru (see [8]). They proved that the IVP of the Navier-Stokes equations is well-posed in BMO^{-1}. This is the largest function space in which well-posedness of the IVP of the Navier-Stokes equations has been established. Note that though not rigorously proved, it is commonly believed that similar results hold true for the initial-boundary value problem of the Navier-Stokes equations with a periodic boundary condition.

All the function spaces in which the IVP of the Navier-Stokes are known to be well-posed are either critical or subcritical spaces. To explain these concepts let us first consider the scaling invariance property of these equations. For any $\lambda > 0$ we denote

$$\vec{u}_\lambda(x,t) = \lambda \vec{u}(\lambda x, \lambda^2 t), \quad P_\lambda(x,t) = \lambda P(\lambda^2 x, \lambda^2 t).$$

It can be easily seen that if $(\vec{u}, P)$ is a solution of the Navier-Stokes equations, then so is $(\vec{u}_\lambda, P_\lambda)$ for any $\lambda > 0$. Hence we introduce the scaling $\varphi_\lambda(x) = \lambda\varphi(\lambda x)$. A function space X on $\mathbb{R}^N$ is respectively called critical, subcritical and supercritical to the Navier-Stokes equations if its corresponding homogeneous version space $\dot{X}$ respectively satisfies the properties $\|\varphi_\lambda\|_{\dot{X}} = \|\varphi\|_{\dot{X}}$, $\|\varphi_\lambda\|_{\dot{X}} = \lambda^\mu \|\varphi\|_{\dot{X}}$ and $\|\varphi_\lambda\|_{\dot{X}} = \lambda^{-\mu}\|\varphi\|_{\dot{X}}$ for some $\mu > 0$ and any $\lambda > 0$ and $\varphi \in \dot{X}$. Though not proved yet, it is strongly conjectured that in supercritical spaces the IVP of the Navier-Stokes equations are usually ill-posed. Hence, existence of solutions for the IVP of the Navier-Stokes equations for initial data in supercritical spaces are not known except for the $L^2(\mathbb{R}^N)$ space (which is supercritical for $N \geq 3$ but critical for $N = 2$) and sums of this space with some well-posedness spaces (cf. [3], [9] and the recent work of the second author [6]).

Inspired by the interesting works of Burq and Tzvetkov [1, 2] on the random-data initial-boundary value problems of the nonlinear wave equations, in this paper we want to construct solutions for the problem (1.1) for a random class of initial data in $\mathcal{H}^s(\mathbb{T}^N)$ with $-1 < s < 0$, where $\mathcal{H}^s(\mathbb{T}^N)$ is the classical L^2-type Sobolev space of index s on the torus $\mathbb{T}^N$, equipped with the norm $\|\vec{u}\|_{\mathcal{H}^s(\mathbb{T}^N)} = \|(1-\Delta)^{\frac{s}{2}}\vec{u}\|_{L^2(\mathbb{T}^N)}$. Note that for any $s < 0$, $\mathcal{H}^s(\mathbb{T}^N)$ is a supercritical space, so that it cannot be embedded into any critical or subcritical spaces (if a space X is embedded into a critical or a subcritical space Y, then X itself must be either a critical or a subcritical space), and cannot be decomposed into a sum of the $L^2(\mathbb{R}^N)$ with a critical or a subcritical space either. Hence existence of solution for initial data in this space can not be obtained from known results. We must study this problem with new ideas.

Let us now turn to present the main results of this paper. For this purpose we first introduce some essential concepts and notations.

First, for a Banach space X of N-vector functions or N-vector distributions on $\mathbb{T}^N$, we use the notation X_σ to denote the corresponding closed subspace of X consisting of all divergence-free elements in X, i.e., $\vec{u} \in X_\sigma$ if and only if $\vec{u} \in X$ and $\nabla \cdot \vec{u} = 0$.

Let $(\Omega, \mathcal{A}, p)$ be a probability space. We consider a sequence of independent and identically distributed Gaussian random variables $(g_n(\omega))_{n\geq 0}$, i.e., $g_n \in \mathcal{N}(0,1)$. Let $(\lambda_n^2)_{n\geq 0}$ be the sequence of eigenvalues of $-\Delta$ on $\mathbb{T}^N$, ordered increasingly and in their multiplicities and we denote by $(e_n)_{n\geq 0} \subseteq C^\infty(\mathbb{T}^N)$ the corresponding sequence of eigenfunctions, normalized so that they form an orthonormal basis of $L^2(\mathbb{T}^N)$. For any $f \in \mathcal{H}^s(\mathbb{T}^N)$, we have

$$f(x) = \sum_{n\geq 0} \alpha_n e_n(x), \quad \alpha_n \in \mathbb{R}, \quad \|f\|^2_{\mathcal{H}^s} \approx \sum_{n\geq 0} |\alpha_n|^2 (1 + |\lambda_n|^2)^s \tag{1.2}$$

We consider the map

$$\omega \longmapsto f^\omega(x) = \sum_{n\geq 0} \alpha_n g_n(\omega) e_n(x) \tag{1.3}$$

from Ω to $\mathcal{H}^s(\mathbb{T}^N)$ (equipped with the Borel sigma algebra). Then f^ω belongs to $L^2(\Omega; \mathcal{H}^s(\mathbb{T}^N))$ (cf. [1, 2]) and the map (1.3) is measurable. The $\mathcal{H}^s(\mathbb{T}^N)$-valued random variable f^ω is called the *randomization* of f. Similarly, we define the randomization of a vector valued function $\vec{f} = (f_1, \dots, f_N)$ by

$$\vec{f^\omega}(x) = \sum_{n\geq 0} (\alpha_n^1, \cdots, \alpha_n^N) e_n(x) g_n(\omega), \tag{1.4}$$

where

$$f_1 = \sum_{n\geq 0} \alpha_n^1 e_n(x), \ \cdots, f_N = \sum_{n\geq 0} \alpha_n^N e_n(x).$$

For $1 \leq q \leq \infty$, $1 \leq m < \infty$ and $\delta > 0$, we introduce the following notation: For any measurable function $g(t,x)$ on $[0,T] \times \mathbb{T}^N$,

$$\|g\|_{L^m_{\delta;T} L^q_x} = \Big(\int_0^T \|t^\delta g(t,\cdot)\|^m_{L^q(\mathbb{T}^N)} dt \Big)^{\frac{1}{m}}. \tag{1.5}$$

The main results of this paper are as follows:

Theorem 1.1. *Let $N = 2, 3$ and $-1 + \frac{N}{4} < s < 0$. Given $\vec{f} \in \mathcal{H}^s_\sigma(\mathbb{T}^N)$, let $\vec{f^\omega} \in L^2(\Omega; \mathcal{H}^s_\sigma(\mathbb{T}^N))$ be the randomization of $\vec{f}$ given by* (1.4)*. Then there is an event $\Sigma \subseteq \Omega$ with $p(\Sigma) = 1$, such that for any $\omega \in \Sigma$, there exists a corresponding $T_\omega > 0$ such that the problem* (1.1) *with $u_0 = \vec{f^\omega}$ has a unique solution $\vec{u}$ for $0 \leq t \leq T_\omega$, satisfying*

$$\vec{u} - e^{t\Delta} \vec{f^\omega} \in C([0, T_\omega]; \mathcal{H}^s_\sigma(\mathbb{T}^N)) \bigcap L^m_{\delta; T_\omega} L^4_{x,\sigma} \bigcap L^{\frac{8}{4-N}}_{T_\omega} L^4_{x,\sigma}, \tag{1.6}$$

where $\frac{8}{4-N} < m \leq \frac{16}{4-N}$ and $\delta = \frac{4-N}{8} - \frac{1}{m}$. More precisely, there exist constants $c_1, c_2 > 0$ and for every $0 < T \leq 1$ an event Ω_T with the property

$$p(\Omega_T) \geq 1 - c_1 e^{-c_2 \|f\|^{-2}_{\mathcal{H}^s} T^{-2\varrho}},$$

where $\varrho = \min\{\frac{1}{m}, \frac{s}{2} + \frac{4-N}{8}\}$, such that for every $\omega \in \Omega_T$, the problem (1.1) *with $u_0 = \vec{f^\omega}$ has a unique solution for $0 \leq t \leq T$ satisfying a similar property as* (1.6) (*with T_ω replaced by T*)*.*

Theorem 1.2. *Let $N = 2,3$ and $-1 < s \leq -1 + \frac{N}{4}$. Given $\vec{f} \in \mathcal{H}^s_\sigma(\mathbb{T}^N)$, let $\vec{f^\omega} \in L^2(\Omega; \mathcal{H}^s_\sigma(\mathbb{T}^N))$ be the randomization of $\vec{f}$ given by (1.4). Then there is an event $\Sigma \subseteq \Omega$ with $p(\Sigma) = 1$, such that for any $\omega \in \Sigma$, there exists a corresponding $T_\omega > 0$ such that the problem (1.1) with $u_0 = \vec{f^\omega}$ has a unique solution $\vec{u}$ for $0 \leq t \leq T_\omega$, satisfying*

$$\vec{u} - e^{t\Delta}\vec{f^\omega} \in L^{\frac{4}{1-s}}_{T_\omega} L^{\frac{2N}{1+s}}_{x,\sigma}. \tag{1.7}$$

More precisely, there exist constants $c_1, c_2 > 0$ and for every $0 < T \leq 1$ an event Ω'_T with the property

$$p(\Omega'_T) \geq 1 - c_1 e^{-c_2\lambda^2\|f\|^{-2}_{\mathcal{H}^s} T^{-\frac{1+s}{2}}},$$

such that for every $\omega \in \Omega'_T$, the problem (1.1) with $u_0 = \vec{f^\omega}$ has a unique solution for $0 \leq t \leq T$ satisfying a similar property as (1.7) (with T_ω replaced by T).

Remark. First, similarly as in [1] we can prove that for any $s \in \mathbb{R}$ and $\epsilon > 0$, if $\vec{f} \in \mathcal{H}^s_\sigma(\mathbb{T}^N)$ is such that $\vec{f} \notin \mathcal{H}^{s+\epsilon}_\sigma(\mathbb{T}^N)$ then for almost all $\omega \in \Omega$, $\vec{f^\omega} \in \mathcal{H}^s_\sigma(\mathbb{T}^N)$ but $\vec{f^\omega} \notin \mathcal{H}^{s+\epsilon}_\sigma(\mathbb{T}^N)$. Hence the randomization has no regularizing effect. Next, though the solution ensured by Theorems 1.1 and 1.2 are local solutions, the initial data that we consider can be large, so that these results are not small data results. Thirdly, note that the solution satisfies the relation $\nabla \cdot (\vec{u} - e^{t\Delta}\vec{f^\omega}) = 0$. We shall not repeat this fact later on. Finally, though our discussion is only made for the cases $N = 2, 3$, from the proof of Theorem 1.2 we easily see that this result (Theorem 1.2) can be straightforwardly extended to all the cases $N \geq 4$ and $-1 < s \leq 0$.

Later on we shall use C to denote constant which depends only on N and may change from line to line. We also use c to denote constant which depends on N and might depends on s and may change from line to line. We use the notations $A \sim B$ and $A \lesssim B$ stand for $C^{-1}B \leq A \leq CB$ and $A \leq CB$, respectively. The notations L^r_ω, L^p_T and L^q_x stand for $L^r(\Omega)$, $L^p(0,T)$ and $L^q(\mathbb{T}^N)$, respectively, whereas we denote $L^q_\sigma = L^q_\sigma(\mathbb{T}^N)$ and $L^p_T L^q_\sigma = L^p(0,T; L^q_\sigma(\mathbb{T}^N))$.

The rest part is organized as follows. In Section 2 we make some preliminary discussions. In Section 3 we give the proofs of Theorems 1.1 and 1.2.

2 Preliminaries

In this section we make some preliminary discussion. Let $(\lambda_n)_{n\geq 0}$ be the increasing sequence of all eigenvalues of the minus Laplace $-\Delta$ on the torus $\mathbb{T}^N$, where multiple eigenvalues are counted in their multiplicities. Let $(e_n)_{n\geq 0}$ be the corresponding sequence of eigenfunctions suitably chosen such that they form a normalized orthogonal basis of $L^2(\mathbb{T}^N)$. Recall that in the case $N = 1$ we have

$$\lambda_0 = 0, \quad \lambda_{2k-1} = \lambda_{2k} = 4k^2\pi^2, \quad k = 1, 2, \cdots,$$

and

$$e_0(x) = 1, \quad e_{2k-1}(x) = \sqrt{2}\cos(2k\pi x), \quad e_{2k}(x) = \sqrt{2}\sin(2k\pi x), \quad k = 1, 2, \cdots,$$

so that

$$\|e_n\|_{L^2(\mathbb{T}^1)} = 1, \quad \|e_n\|_{L^\infty(\mathbb{T}^1)} \le \sqrt{2}, \quad n = 0, 1, 2, \cdots.$$

Since $L^2(\mathbb{T}^N)$ is the N-tensor product of $L^2(\mathbb{T}^1)$, the above inequalities still hold true for general N when $\sqrt{2}$ is replaced by $(\sqrt{2})^N$. Hence, we have the following fundamental estimate:

Lemma 2.1. *For any $q \in [2, \infty]$ we have*

$$\|e_n\|_{L^q(\mathbb{T}^N)} \lesssim 1. \tag{2.1}$$

Next we quote the following preliminary result from [1] (see Lemma 3.1 there):

Lemma 2.2. *Let $(g_n(\omega))_{n\ge 0}$ be a sequence of Gaussian random variables. For any $r \ge 2$ and $(c_n)_{n\ge 0} \in l^2$, we have*

$$\|\sum_{n\ge 0} c_n g_n(\omega)\|_{L^r(\Omega)} \lesssim \sqrt{r}(\sum_{n\ge 0} |c_n|^2)^{1/2}. \tag{2.2}$$

In what follows we derive some new estimates. Recall that $N = 2, 3$.

Lemma 2.3. *Let $-1 < s < 0$ and $0 < T \le 1$. For $f \in \mathcal{H}^s(\mathbb{T}^N)$, let f^ω be the randomization of f given by (1.3). Then we have the following assertions:*

(i) If $-1 < s \le -1 + \frac{N}{4}$, then for any $r \ge \frac{2N}{1+s}$ we have

$$\|e^{t\Delta} f^\omega\|_{L^r_\omega L_T^{\frac{4}{1-s}} L_x^{\frac{2N}{1+s}}} \lesssim \sqrt{r} T^{\frac{1+s}{4}} \|f\|_{\mathcal{H}^s}. \tag{2.3}$$

As a consequence, if we set $E_{\lambda,f,T;1} = \{\omega \in \Omega : \|e^{t\Delta} f^\omega\|_{L_T^{\frac{4}{1-s}} L_x^{\frac{2N}{1+s}}} \ge \lambda\}$, then there exist $c_1, c_2 > 0$ such that for all $\lambda > 0$ and $f \in \mathcal{H}^s(\mathbb{T}^N)$,

$$p(E_{\lambda,f,T;1}) \le c_1 e^{-c_2 \lambda^2 \|f\|_{\mathcal{H}^s}^{-2} T^{-\frac{1+s}{2}}}. \tag{2.4}$$

(ii) If $-1 + \frac{N}{4} < s < 0$, then for any $r \ge \frac{8}{4-N}$ we have

$$\|e^{t\Delta} f^\omega\|_{L^r_\omega L_T^{\frac{8}{4-N}} L_x^4} \lesssim \sqrt{r} T^{\frac{s}{2} + \frac{4-N}{8}} \|f\|_{\mathcal{H}^s}. \tag{2.5}$$

As a consequence, if we set $E_{\lambda,f,T;2} = \{\omega \in \Omega : \|e^{t\Delta} f^\omega\|_{L_T^{\frac{8}{4-N}} L_x^4} \ge \lambda\}$, then there exist $c_1, c_2 > 0$ such that for all $\lambda > 0$ and $f \in \mathcal{H}^s(\mathbb{T}^N)$,

$$p(E_{\lambda,f,T;2}) \le c_1 e^{-c_2 \lambda^2 \|f\|_{\mathcal{H}^s}^{2-2} T^{-s + \frac{N-4}{4}}}. \tag{2.6}$$

Proof. Given $f \in \mathcal{H}^s(\mathbb{T}^N)$, let $f = \sum_{n\ge 0} \alpha_n e_n$. Then by (1.4) we have

$$e^{t\Delta} f^\omega = \sum_{n\ge 0} e^{-t\lambda_n^2} \alpha_n g_n(\omega) e_n(x).$$

We first assume that $-1 < s \le -1 + \frac{N}{4}$. In this case we have $\frac{2N}{1+s} \ge \frac{4}{1-s} > 2$. Thus for any $r \ge \frac{2N}{1+s}$, by using the Minkowski inequality and Lemmas 2.1 $\sim$ 2.2 we see that

$$
\begin{aligned}
\|e^{t\Delta} f^\omega\|^2_{L^r_\omega L_T^{\frac{4}{1-s}} L_x^{\frac{2N}{1+s}}} &\lesssim \|e^{t\Delta} f^\omega\|^2_{L_T^{\frac{4}{1-s}} L_x^{\frac{2N}{1+s}} L^r_\omega} \\
&\lesssim r \| \sum_{n\ge 0} e^{-2t\lambda_n^2} |\alpha_n|^2 |e_n(x)|^2 \|_{L_T^{\frac{2}{1-s}} L_x^{\frac{N}{1+s}}} \\
&\lesssim r \| \sum_{n\ge 0} |\alpha_n|^2 e^{-2t\lambda_n^2} \|e_n\|^2_{L^{\frac{2N}{1+s}}} \|_{L_T^{\frac{2}{1-s}}} \\
&\lesssim r \| \sum_{n\ge 0} |\alpha_n|^2 e^{-2t\lambda_n^2} \|_{L_T^{\frac{2}{1-s}}} \\
&\lesssim r \Big(|\alpha_0|^2 T^{\frac{1-s}{2}} + \sum_{n\ge 1} |\alpha_n|^2 \|e^{-2t\lambda_n^2}\|_{L_T^{\frac{1}{|s|}}} T^{\frac{1+s}{2}} \Big) \\
&\lesssim r \Big(|\alpha_0|^2 T^{\frac{1-s}{2}} + \sum_{n\ge 1} |\alpha_n|^2 \lambda_n^{2s} T^{\frac{1+s}{2}} \Big) \\
&\lesssim r T^{\min\{\frac{1-s}{2}, \frac{1+s}{2}\}} \sum_{n\ge 0} |\alpha_n|^2 (1+\lambda_n^2)^s \\
&\lesssim r T^{\frac{1+s}{2}} \|f\|^2_{\mathcal{H}^s}.
\end{aligned} \tag{2.7}
$$

Hence (2.3) follows. From (2.3) and the Bienaymé–Tchebychev inequality, it follows that there exists constant $C > 0$ (independent of f, T and r) such that

$$
p(\omega \in \Omega : \|e^{t\Delta} f^\omega\|_{L_T^{\frac{4}{1-s}} L_x^{\frac{2N}{1+s}}} \ge \lambda) \le (C\sqrt{r}\lambda^{-1} \|f\|_{\mathcal{H}^s} T^{\frac{1+s}{4}})^r. \tag{2.8}
$$

From this estimate we can easily obtain (2.4). Indeed, if $\lambda > 0$ is so small such that

$$
\Big(\frac{\lambda}{Ce\|f\|_{\mathcal{H}^s} T^{\frac{1+s}{4}}} \Big)^2 \le \frac{2N}{1+s},
$$

then (2.4) follows from the facts that $p(E) \le 1$ for any event E and that the function e^{-x^2} has positive lower bound for x in any bounded set, otherwise we choose

$$
r := \Big(\frac{\lambda}{Ce\|f\|_{\mathcal{H}^s} T^{\frac{1+s}{4}}} \Big)^2 > \frac{2N}{1+s}.
$$

Then from (2.8) we see that

$$
p(E_{\lambda,f,T;1}) \le e^{-(Ce)^{-2}\lambda^2 \|f\|^{-2}_{\mathcal{H}^s} T^{-\frac{1+s}{2}}},
$$

by which (2.4) follows. This proves the assertion (i).

Next we assume that $-1 + \frac{N}{4} < s < 0$. In this case, again by the Minkowski inequality and Lemmas 2.1 $\sim$ 2.2 we see that for any $r \ge \frac{8}{4-N} \ge 4$,

$$
\|e^{t\Delta} f^\omega\|^2_{L^r_\omega L_T^{\frac{8}{4-N}} L_x^4} \lesssim \|e^{t\Delta} f^\omega\|^2_{L_T^{\frac{8}{4-N}} L_x^4 L^r_\omega}
$$

$$
\begin{aligned}
&\lesssim r\Big\|\sum_{n\geq 0} e^{-2t\lambda_n^2}|\alpha_n|^2|e_n(x)|^2\Big\|_{L_T^{\frac{4}{4-N}}L_x^2}\\
&\lesssim r\Big\|\sum_{n\geq 0} e^{-2t\lambda_n^2}|\alpha_n|^2\|e_n\|_{L_x^4}^2\Big\|_{L_T^{\frac{4}{4-N}}}\\
&\lesssim r\Big\|\sum_{n\geq 0} e^{-2t\lambda_n^2}|\alpha_n|^2\Big\|_{L_T^{\frac{4}{4-N}}}\\
&\lesssim r\Big(|\alpha_0|^2T^{\frac{4-N}{4}}+\sum_{n\geq 1}|\alpha_n|^2\|e^{-2t\lambda_n^2}\|_{L_T^{\frac{1}{|s|}}}T^{s+\frac{4-N}{4}}\Big)\\
&\lesssim rT^{\min\{\frac{4-N}{4},s+\frac{4-N}{4}\}}\Big(|\alpha_0|^2+\sum_{n\geq 1}|\alpha_n|^2\lambda_n^{2s}\Big)\\
&\lesssim rT^{s+\frac{4-N}{4}}\|f\|_{\mathcal{H}^s}^2.
\end{aligned}
$$

Hence (2.5) follows. From (2.5) and a similar argument as before we obtain (2.6). This proves the assertion *(ii)* and completes the proof of Lemma 2.3. $\square$

Lemma 2.4. *Let $0 < T \leq 1$, $-1+\frac{N}{4} < s < 0 < \delta \leq \frac{4-N}{16}$, $\frac{8}{4-N} < m \leq \frac{16}{4-N}$ and $\frac{1}{m}+\delta = \frac{4-N}{8}$. For $f \in \mathcal{H}^s(\mathbb{T}^N)$, let f^ω be the randomization of f given by (1.3). Then for any $r \geq m$ we have*

$$
\|e^{t\Delta}f^\omega\|_{L_\omega^r L_{\delta;T}^m L_x^4} \lesssim \sqrt{r}T^{\varrho}\|f\|_{\mathcal{H}^s}, \tag{2.9}
$$

where $\varrho = \min\{\frac{1}{m}, \frac{s}{2}+\frac{4-N}{8}\}$. Consequently, if we set $E_{\lambda,f,T;3} = \{\omega \in \Omega : \|e^{t\Delta}f^\omega\|_{L_{\delta;T}^m L_x^4} \geq \lambda\}$, then there exist $c_1, c_2 > 0$ such that for all $\lambda > 0$ and $f \in \mathcal{H}^s(\mathbb{T}^N)$,

$$
p(E_{\lambda,f,T;3}) \leq c_1 e^{-c_2\lambda^2\|f\|_{\mathcal{H}^s}^{-2}T^{-2\varrho}}. \tag{2.10}
$$

Proof. Similarly as before, by writing $f = \sum_{n\geq 0}\alpha_n e_n$, we have

$$
t^\delta e^{t\Delta}f^\omega = \sum_{n\geq 0} t^\delta e^{-t\lambda_n^2}\alpha_n g_n(\omega)e_n(x).
$$

Since $r \geq m > \frac{8}{4-N} \geq 4$, by the Minkowski inequality and Lemmas 2.1 $\sim$ 2.2 we have

$$
\begin{aligned}
\|t^\delta e^{t\Delta}f^\omega\|_{L_\omega^r L_T^m L_x^4}^2 &\lesssim r\|t^\delta e^{t\Delta}f^\omega\|_{L_T^m L_x^4 L_\omega^r}^2\\
&\lesssim r\Big\|\sum_{n\geq 0} t^{2\delta}e^{-2t\lambda_n^2}|\alpha_n|^2|e_n(x)|^2\Big\|_{L_T^{\frac{m}{2}}L_x^2}\\
&\lesssim r\Big\|\sum_{n\geq 0}|\alpha_n|^2t^{2\delta}e^{-2t\lambda_n^2}\|e_n\|_{L_x^4}^2\Big\|_{L_T^{\frac{m}{2}}}\\
&\lesssim r\Big\|\sum_{n\geq 0}|\alpha_n|^2t^{2\delta}e^{-2t\lambda_n^2}\Big\|_{L_T^{\frac{m}{2}}}\\
&\lesssim r\Big(T^{\frac{2}{m}}|\alpha_0|^2+\sum_{n\geq 1}|\alpha_n|^2\|t^{2\delta}e^{-2t\lambda_n^2}\|_{L_T^{\frac{m}{2}}}\Big).
\end{aligned} \tag{2.11}
$$

To estimates $\|t^{2\delta}e^{-2t\lambda_n^2}\|_{L_T^{\frac{m}{2}}}$ (for $n \geq 1$), we consider the two different cases $-1+\frac{N}{4} \leq s \leq -\frac{2}{m}$ and $-\frac{2}{m} < s < 0$ separately. In the first case we have

$$\begin{aligned}\|t^{2\delta}e^{-2t\lambda_n^2}\|_{L_T^{\frac{m}{2}}} &\lesssim T^{s+\frac{2}{m}+2\delta}\|t^{-s-\frac{2}{m}}e^{-2t\lambda_n^2}\|_{L_T^{\frac{m}{2}}} \\ &\lesssim T^{s+\frac{2}{m}+2\delta}\lambda_n^{2s} \lesssim T^{s+\frac{4-N}{4}}(1+\lambda_n^2)^s. \end{aligned} \tag{2.12}$$

In the second case we have

$$\|t^{2\delta}e^{-2t\lambda_n^2}\|_{L_T^{\frac{m}{2}}} \lesssim T^{2\delta+\frac{2}{m}+s}\|e^{-2t\lambda_n^2}\|_{L_T^{\frac{1}{|s|}}} \lesssim T^{s+\frac{4-N}{4}}(1+\lambda_n^2)^s. \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), we see that

$$\|e^{t\Delta}f^\omega\|_{L_\omega^r L_{\delta;T}^m L_x^4} \lesssim \sqrt{r}T^{\min\{\frac{1}{m},\frac{s}{2}+\frac{4-N}{8}\}}\|f\|_{\mathcal{H}^s}.$$

This proves (2.9). Having proved (2.9), (2.10) follows from a similar argument as before. This completes the proof of Lemma 2.4. $\square$

3 Proofs of Theorems 1.1 and 1.2

In this section we give the proofs of Theorems 1.1 and 1.2. To this end, we convert the problem (1.1) with $u_0 = \vec{f}^\omega$ into the following equivalent integral equation:

$$\vec{u}(t,\cdot) = \vec{u}_{\vec{f}}^\omega - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(\vec{u}\otimes\vec{u})(\tau,\cdot)d\tau, \tag{3.1}$$

where $\vec{u}_{\vec{f}}^\omega = e^{t\Delta}\vec{f}^\omega$, $\otimes$ denotes tensor product between vectors, and $\mathbb{P}$ is the Helmholtz-Weyl projection operator, i.e., $\mathbb{P} = I + \nabla(-\Delta)^{-1}\nabla$. We make the unknown variable transformation $\vec{u} \mapsto \vec{v}$ by letting $\vec{u} = \vec{u}_{\vec{f}}^\omega + \vec{v}$. Then the above integral equation is transformed into the following equivalent one:

$$\vec{v}(t,\cdot) = -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot[(\vec{u}_{\vec{f}}^\omega+\vec{v})\otimes(\vec{u}_{\vec{f}}^\omega+\vec{v})](\tau,\cdot)d\tau. \tag{3.2}$$

We shall use this equivalent form of the integral equation (3.2) to prove Theorems 1.1 and 1.2. In what follows, we denote by $K_{\vec{f}}^\omega$ the following operator:

$$K_{\vec{f}}^\omega : \vec{v} \mapsto -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot[(\vec{u}_{\vec{f}}^\omega+\vec{v})\otimes(\vec{u}_{\vec{f}}^\omega+\vec{v})](\tau,\cdot)d\tau. \tag{3.3}$$

Proof of Theorem 1.1: Let m and δ be as in Theorem 1.1, i.e., $\frac{16}{4-N} \geq m > \frac{8}{4-N}$ and $\delta = \frac{4-N}{8} - \frac{1}{m}$. Note that $0 < \delta \leq \frac{4-N}{16}$. Given $T > 0$, let $X = X_T$ be the space

$$X = C\Big([0,T];\mathcal{H}_\sigma^{\frac{N}{2}-1}(\mathbb{T}^N)\Big)\bigcap L_{\delta;T}^m L_{x,\sigma}^4 \bigcap L_T^{\frac{8}{4-N}}L_{x,\sigma}^4$$

with norm

$$\|u\|_X = \|u\|_{L^\infty_T \mathcal{H}^{(N-2)/2}_x} + \|u\|_{L^m_{\delta;T} L^4_x} + \|u\|_{L^{8/(4-N)}_T L^4_x} \quad \text{for } \ u \in X.$$

It is clear that $(X, \|\cdot\|_X)$ is a Banach space. We introduce the event

$$E_{\lambda,\vec{f},T} = \{\omega \in \Omega : \|\vec{u}^\omega_{\vec{f}}\|_{L^m_{\delta;T} L^4_x} + \|\vec{u}^\omega_{\vec{f}}\|_{L_T^{\frac{8}{4-N}} L^4_x} \geq \lambda\}. \tag{3.4}$$

In what follows we prove that for any $\omega \in \Omega \backslash E_{\lambda,\vec{f},T}$, the operator $K^\omega_{\vec{f}}$ is well-defined in X, maps X into itself, and is a contraction mapping when restricted to some closed ball in X provided λ is chosen sufficiently small. To this end, for a given $\vec{v} \in X$, we estimate the three norms $\|K^\omega_{\vec{f}}(\vec{v})\|_{L^\infty_T \mathcal{H}^{(N-2)/2}_x}$, $\|K^\omega_{\vec{f}}(\vec{v})\|_{L^m_{\delta;T} L^4_x}$ and $\|K^\omega_{\vec{f}}(\vec{v})\|_{L^{8/(4-N)}_T L^4_x}$ one by one.

Step 1: Estimate of $\|K^\omega_{\vec{f}}(\vec{v})\|_{L^\infty_T \mathcal{H}^{(N-2)/2}_x}$. To estimate this norm, we first prove that for any $\vec{u}, \vec{v} \in L^m_{\delta;T} L^4_{x,\sigma}$,

$$\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{v})(\tau) d\tau \|_{\mathcal{H}^{\frac{N-2}{2}}} \lesssim \|\vec{u}\|_{L^m_{\delta;T} L^4_x} \|\vec{v}\|_{L^m_{\delta;T} L^4_x}. \tag{3.5}$$

In fact, we have

$$\begin{aligned}
&\|\int_0^t e^{(t-\tau)\Delta} \mathbb{P}\nabla \cdot (\vec{v} \otimes \vec{v})(\tau) d\tau \|_{\mathcal{H}^{\frac{N-2}{2}}} \lesssim \int_0^t (t-\tau)^{-\frac{1}{2}} \Big(1 + (t-\tau)^{-\frac{N-2}{4}}\Big) \| |\vec{u}(\tau)| |\vec{v}(\tau)| \|_{L^2_x} d\tau \\
&\lesssim \Big\{ \int_0^t \Big[(t-\tau)^{-\frac{1}{2}} \Big(1 + (t-\tau)^{-\frac{N-2}{4}}\Big) \tau^{-2\delta} \Big]^q d\tau \Big\}^{\frac{1}{q}} \|\vec{u}\|_{L^m_{\delta;T} L^4_x} \|\vec{v}\|_{L^m_{\delta;T} L^4_x} \\
&\lesssim \|\vec{u}\|_{L^m_{\delta;T} L^4_x} \|\vec{v}\|_{L^m_{\delta;T} L^4_x},
\end{aligned}$$

where $\frac{1}{q} = 1 - \frac{2}{m}$ (note that $q > 1$ and $(\frac{1}{2} + \frac{N-2}{4} + 2\delta)q = 1$). This proves (3.5). Since $\vec{u}^\omega_{\vec{f}} \in L^m_{\delta;T} L^4_{x,\sigma}$ and $\|\vec{u}^\omega_{\vec{f}}\|_{L^m_{\delta;T} L^4_x} \leq \lambda$ for $\omega \in \Omega \backslash E_{\lambda,\vec{f},T}$, by (3.5) we immediately obtain

$$\|K^\omega_{\vec{f}}(\vec{v})\|_{L^\infty_T \mathcal{H}^{(N-2)/2}_x} \leq C(\lambda^2 + \|\vec{v}\|^2_{L^m_{\delta;T} L^4_x}). \tag{3.6}$$

Step 2: Estimate of $\|K^\omega_{\vec{f}}(\vec{v})\|_{L^m_{\delta;T} L^4_x}$. We first prove that for any $\vec{u}, \vec{v} \in L^m_{\delta;T} L^4_{x,\sigma} \cap L_T^{\frac{8}{4-N}} L^4_{x,\sigma}$,

$$\begin{aligned}
\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{v})(\tau) d\tau \|_{L^m_{\delta;T} L^4_x} \lesssim \min\{ &(\|\vec{u}\|_{L^m_{\delta;T} L^4_x} + \|\vec{u}\|_{L_T^{\frac{8}{4-N}} L^4_x}) \|\vec{v}\|_{L_T^{\frac{8}{4-N}} L^4_x}, \\
&(\|\vec{v}\|_{L^m_{\delta;T} L^4_x} + \|\vec{v}\|_{L_T^{\frac{8}{4-N}} L^4_x}) \|\vec{u}\|_{L_T^{\frac{8}{4-N}} L^4_x} \}.
\end{aligned} \tag{3.7}$$

In fact, we have

$$\|t^\delta \int_0^t e^{(t-\tau)\Delta} \mathbb{P}\nabla \cdot (\vec{u} \otimes \vec{v})(\tau) d\tau \|_{L^m_T L^4_x} \lesssim \| \int_0^t t^\delta (t-\tau)^{-\frac{1}{2} - \frac{N}{8}} \| |\vec{u}(\tau)| |\vec{v}(\tau)| \|_{L^2_x} d\tau \|_{L^m_T}$$

$$
\begin{aligned}
&\lesssim \| \int_{\frac{t}{2}}^{t} t^{\delta}(t-\tau)^{-\frac{N+4}{8}} \| |\vec{u}(\tau)||\vec{v}(\tau)| \|_{L_x^2} d\tau \|_{L_T^m} \\
&\quad + \| \int_0^{\frac{t}{2}} t^{\delta}(t-\tau)^{-\frac{N+4}{8}} \| |\vec{u}(\tau)||\vec{v}(\tau)| \|_{L_x^2} d\tau \|_{L_T^m} \\
&:= I_1 + I_2.
\end{aligned}
$$

By using the Hardy-Littlewood-Sobolev and the Hölder inequalities, we have

$$
\begin{aligned}
I_1 &\lesssim \| \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{N+4}{8}} \| \tau^{\delta} |\vec{u}(\tau)||\vec{v}(\tau)| \|_{L_x^2} d\tau \|_{L_T^m} \\
&\lesssim \min\{ \|\vec{u}\|_{L^m_{\delta;T} L^4_x} \|\vec{v}\|_{L_T^{\frac{8}{4-N}} L_x^4}, \|\vec{u}\|_{L_T^{\frac{8}{4-N}} L_x^4} \|\vec{v}\|_{L^m_{\delta;T} L^4_x} \}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &\lesssim \| \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{N+4}{8}+\delta} \| |\vec{u}(\tau)||\vec{v}(\tau)| \|_{L_x^2} d\tau \|_{L_T^m} \\
&\lesssim \| \|\vec{v}(\tau)\|_{L_x^4} \|\vec{u}(\tau)\|_{L_x^4} \|_{L_T^{\frac{4}{4-N}}} \\
&\lesssim \|\vec{u}\|_{L_T^{\frac{8}{4-N}} L_x^4} \|\vec{v}\|_{L_T^{\frac{8}{4-N}} L_x^4}.
\end{aligned}
$$

This proves (3.7). Since $\vec{u}^{\omega}_{\vec{f}} \in L^m_{\delta;T} L^4_{x,\sigma} \bigcap L_T^{\frac{8}{4-N}} L^4_{x,\sigma}$ and $\|\vec{u}^{\omega}_{\vec{f}}\|_{L^m_{\delta;T} L^4_x} + \|\vec{u}^{\omega}_{\vec{f}}\|_{L_T^{\frac{8}{4-N}} L_x^4} \leq \lambda$ for $\omega \in \Omega \backslash E_{\lambda,\vec{f},T}$, by (3.7) we immediately obtain

$$
\|K^{\omega}_{\vec{f}}(\vec{v})\|_{L^m_{\delta;T} L^4_x} \leq C(\lambda^2 + \|\vec{v}\|^2_{L^m_{\delta;T} L^4_x}). \tag{3.8}
$$

Step 3: Estimate of $\|K^{\omega}_{\vec{f}}(\vec{v})\|_{L_T^{\frac{8}{4-N}} L_x^4}$. Similarly as before, for any $\vec{u}, \vec{v} \in L_T^{\frac{8}{4-N}} L^4_{x,\sigma}$ we have

$$
\begin{aligned}
\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{v})(\tau) d\tau \|_{L_T^{\frac{8}{4-N}} L_x^4} &\lesssim \| \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{8}} \| |\vec{u}(\tau)||\vec{v}(\tau)| \|_{L_x^2} d\tau \|_{L_T^{\frac{8}{4-N}}} \\
&\lesssim \|\vec{u}\|_{L_T^{\frac{8}{4-N}} L_x^4} \|\vec{v}\|_{L_T^{\frac{8}{4-N}} L_x^4}.
\end{aligned}
$$

From this inequality and a similar argument as before we conclude that for any $\omega \in \Omega \backslash E_{\lambda,\vec{f},T}$,

$$
\|K^{\omega}_{\vec{f}}(\vec{v})\|_{L_T^{\frac{8}{4-N}} L_x^4} \leq C(\lambda^2 + \|\vec{v}\|^2_{L_T^{\frac{8}{4-N}} L_x^4}). \tag{3.9}
$$

Combining (3.6), (3.8) and (3.9), we see that $K^{\omega}_{\vec{f}}$ is well defined in X, and

$$
\|K^{\omega}_{\vec{f}}(\vec{v})\|_X \leq C_1 \Big(\lambda^2 + \|\vec{v}\|_X^2 \Big). \tag{3.10}
$$

Moreover, by a similar argument we can also prove that for any $\vec{v}_1, \vec{v}_2 \in X$ and $\omega \in \Omega \backslash E_{\lambda,\vec{f},T}$, there holds

$$
\|K^{\omega}_{\vec{f}}(\vec{v}_1) - K^{\omega}_{\vec{f}}(\vec{v}_2)\|_X \leq C_2 \Big(\lambda + \|\vec{v}_1\|_X + \|\vec{v}_2\|_X \Big) \|\vec{v}_1 - \vec{v}_2\|_X. \tag{3.11}
$$

From (3.10) and (3.11) we easily see that if we set $\lambda_0 = \min\{(2C_1)^{-1}, (6C_2)^{-1}\}$, then for any $0 < \lambda \le \lambda_0$ and $\omega \in \Omega \backslash E_{\lambda,\vec{f},T}$, the operator $K^{\omega}_{\vec{f}}$ maps the closed ball $\overline{B}(0,\lambda)$ in X into itself and is a contraction mapping when restricted to this closed ball, so that it has a unique fixed point in this closed ball.

Now arbitrarily choose a λ in $(0, \lambda_0]$ and fix it. We denote

$$\Omega_T = \Omega \backslash E_{\lambda,\vec{f},T} = \{\omega \in \Omega : \|\vec{u}^{\omega}_{\vec{f}}\|_{L^m_{\delta;T} L^4_{\sigma}(\mathbb{T}^N)} + \|\vec{u}^{\omega}_{\vec{f}}\|_{L_T^{\frac{8}{4-N}} L^4_{\sigma}(\mathbb{T}^N)} < \lambda\}.$$

Let c_1, c_2 and ϱ as in Lemmas 2.3 $\sim$ 2.4. Then by noticing that $E_{\lambda,\vec{f},T} \subseteq E_{\frac{\lambda}{2},\vec{f},T;2} \cup E_{\frac{\lambda}{2},\vec{f},T;3}$ and using these lemmas we see that for any $0 < T \le 1$,

$$p(\Omega_T) = 1 - p(E_{\lambda,\vec{f},T}) \ge 1 - 2c_1 e^{-c_2 (\frac{\lambda}{2})^2 \|f\|^{-2}_{\mathcal{H}^s} T^{-2\varrho}}.$$

Hence the second part of the assertion in Theorem 1.1 is proved. Next, for any $j \in \mathbb{N}$, we choose $T_j := T(j, N, s, \lambda, \|\vec{f}\|_{\mathcal{H}^s}) \in (0,1)$ so small such that

$$c_1 e^{-c_2 (\frac{\lambda}{2})^2 \|\vec{f}\|^{-2}_{\mathcal{H}^s} T_j^{-2\varrho}} \le 2^{-j-1}.$$

Then

$$p(\Omega_{T_j}) \ge 1 - 2c_1 e^{-c_2 (\frac{\lambda}{2})^2 \|f\|^{-2}_{\mathcal{H}^s} T_j^{-2\varrho}} \ge 1 - 2^{-j}.$$

We now let $\Sigma = \bigcup_{j \in \mathbb{N}} \Omega_{T_j}$. Clearly, $p(\Sigma) = 1$. Since for any $\omega \in \Sigma$ there exists a corresponding $j = j(\omega) \in \mathbb{N}$ such that $\omega \in \Omega_{T_{j(\omega)}}$, by letting $T_\omega = T_{j(\omega)}$, we see that the first part of the assertion in Theorem 1.1 follows. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2: Let s be as in Theorem 1.2, i.e., $-1 < s \le -1 + \frac{N}{4}$. For given $T > 0$, let Y be the Banach space $L_T^{\frac{4}{1-s}} L_{x,\sigma}^{\frac{2N}{1+s}}$. By using some similar inequalities as before we see that for any $v \in Y$,

$$\begin{aligned} \| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{v})(\tau) d\tau \|_Y &= \| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{v})(\tau) d\tau \|_{L_T^{\frac{4}{1-s}} L_x^{\frac{2N}{1+s}}} \\ &\lesssim \| \int_0^t (t-\tau)^{-\frac{1}{2} - \frac{1+s}{4}} \| |\vec{u}(\tau)| |\vec{v}(\tau)| \|_{L_x^{\frac{N}{1+s}}} d\tau \|_{L_T^{\frac{4}{1-s}}} \\ &\lesssim \| |\vec{u}| |\vec{v}| \|_{L_T^{\frac{2}{1-s}} L_x^{\frac{N}{1+s}}} \lesssim \|\vec{u}\|_Y \|\vec{v}\|_Y. \end{aligned} \tag{3.12}$$

Now set

$$\widetilde{E}_{\lambda,\vec{f},T} = \{\omega \in \Omega : \|\vec{u}^{\omega}_{\vec{f}}\|_Y = \|\vec{u}^{\omega}_{\vec{f}}\|_{L_T^{\frac{4}{1-s}} L_x^{\frac{2N}{1+s}}} \ge \lambda\}.$$

Since $\|\vec{u}^{\omega}_{\vec{f}}\|_Y \le \lambda$ for $\omega \in \Omega \backslash \widetilde{E}_{\lambda,\vec{f},T}$, using (3.12) we easily see that for any $\omega \in \Omega \backslash \widetilde{E}_{\lambda,\vec{f},T}$ and $v, \vec{v}_1, \vec{v}_2 \in Y$ there hold

$$\begin{aligned} \|K^{\omega}_{\vec{f}}(\vec{v})\|_Y &\le C_1 (\lambda^2 + \|\vec{v}\|_Y), \\ \|K^{\omega}_{\vec{f}}(\vec{v_1}) - K^{\omega}_{\vec{f}}(\vec{v_2})\|_Y &\le C_2 \Big(\lambda + \|\vec{v}_1\|_Y + \|\vec{v}_2\|_Y \Big) \|\vec{v}_1 - \vec{v}_2\|_Y. \end{aligned}$$

From these estimates and a similar argument as in the proof of Theorem 1.1, we obtain the desired assertion in Theorem 1.2. This completes the proof of Theorem 1.2.

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